Euler's Gamma Function.

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It is common in mathematics education to first introduce the exponential function a^n as "repeated multiplication", that is $a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}$. Later on in ones mathematics education it is common to learn about the function $e^x = \sum \frac{x^k}{k!}$ which allows for real and complex inputs, and thus generalizes the exponential function to a new domain where the initial definition is nonsensical. It goes with out saying that this is an extremely powerful and useful idea given the heavy mathematical machinery reliant on the theory of the function e^x, for example, Fourier Transforms. The problem of finding simple and smooth curves that generalize functions defined only for integers can fall into a class of problems known as interpolation problems. There is some ambiguity in what an answer to the interpolation problem looks like, and there are several modifications that can be made to the problem statement. We may wish to know if an answer to the interpolation problem is unique in some sense, or if it belongs to a family of smooth curves which all satisfy the interpolation problem. Consider the example $T(n) = \sum\limits_{k \leq n} k$. We know that $T(n) = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$, however when considered as functions of a real number we may not have equality. There are many functions which we may define for integers that initially do not make sense or are uninteresting for real or complex inputs. Sometimes as in the case of $T(n)$, a real input may be reasonable, however the result is discontinuous and thus, uninteresting or displeasing. The function we will study in this report is the factorial function $n! = 1 \cdot 2 \cdot 3 \cdots n$, and specifically the function $\Gamma : \mathbb{C} \to \mathbb{C}$ for which we have $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$. The definition of the factorial function is only reasonable for natural numbers, however it may be useful or interesting to have a sensible definition for the value of $\frac{1}{2}!$ or $i!$. The function Γ is one function that seeks to answer this question in a sensible way.

It appears that the specific problem of interpolating a smooth curve through the points $(n, n!)$ was first seriously considered in the early 1700's by Daniel Bernoulli and Christian Goldbach. Leonard Euler was the first to respond, in the year 1729 he gave the formula

$$n! = \prod_{k=1}^{\infty} \frac{\left(1 + \frac{1}{k}\right)^n}{1 + \frac{n}{k}}. \tag{1}$$

This answer is not entirely satisfactory for many reasons, we may be concerned by when an infinite product has a finite result, or that it is difficult to perform actual computations with infinite products. Nevertheless it is a solution to the interpolation problem, and an interesting one to study since it seems to accept any real or complex input. One can show that at $n = \frac{1}{2}$ the product (1) is connected to the famous Wallis product for π,

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}.$$

In some sense we have found a connection between the factorial function and π, we will later see that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. In the year 1730 Euler gave a formula for $n!$ in the form of an integral,

$$n! = \int_0^1 (-\log s)^n ds = \int_0^1 \log^n \frac{1}{s} ds. \tag{2}$$

This solution is interesting, but note that it is invalid for negative real numbers. Both (1) and (2) mark the beginning of a deep theory of the function Γ.

In modern mathematical writing, the definition of the gamma function is commonly given as

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx. \tag{3}$$

The above integral is a modification of (2) accredited to Adrien Marie Legendre. We can show that (3) converges for $z > 0$. Note that for $z \in [0, 1)$ the integral $\int_0^\infty x^z e^{-x} dx < \int_0^\infty (x+1) e^{-x} dx$ converges. When $z \geq 1$ we can preform integration by parts $[z]$ times to obtain

$$\int_0^{\infty} x^z e^{-z} dx = \left(\prod_{k=0}^{[z]-1} (z-k) \right) \int_0^{\infty} x^{z-[z]} e^{-x} dx.$$

The integral $\int_0^\infty x^{z-[z]} e^{-x} dx$ exists since $z - [z] \in [0, 1)$, thus (3) converges for $z \geq 1$. Also, for $z \in (0, 1)$ we have

$$\int_0^{\infty} x^{-z} e^{-z} dx = \Big[\frac{x^{1-z} e^{-x}}{1-z} \Big]_0^{\infty} + \int_0^{\infty} \frac{x^{1-z} e^{-x}}{1-z} dx = 0 + \int_0^{\infty} \frac{x^{1-z} e^{-x}}{1-z} dx.$$

Since $1-z \in (0,1)$, the integral $\int_0^\infty \frac{x^{1-z}e^{-x}}{1-z}dx = \frac{1}{1-z}\int_0^\infty x^{1-z}e^{-x}dx$ converges and thus (3) converges on positive real inputs as desired.

Continuing in this fashion, we may apply integration by parts and limit rules to discover one key property of Γ,

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty x^z e^{-x} dx \\ &= \big[-x^z e^{-x}\big]_0^\infty + \int_0^\infty z x^{z-1} e^{-x} dx \\ &= z\int_0^\infty x^{z-1}e^{-x}dx \\ &= z\Gamma(z).\end{aligned}$$

Since $\Gamma(1) = \int_0^\infty e^{-x}dx = \big[-e^{-x}\big]_0^\infty = 0-(-1) = 1$, we can use the property $\Gamma(n+1) = n\Gamma(n)$ to inductively verify $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. It is common practice to define gamma such that $\Gamma(n) = (n-1)!$ instead of a definition in which Γ would equal $n!$ directly. Amazingly, on the domain which the functions share there, is equality between $\Gamma(z+1)$, (1) and (2). A natural question one might have is about the uniqueness of Γ, one idea involves a result from complex analysis about uniqueness of analytic continuations of holomorphic functions. Let A be an open subset of $\mathbb{C}$ and let $B \supset A$. Suppose that f is a holomorphic function defined on A and g, h are functions defined on B such that $h = g = f$ for all inputs in A, then $g = h$ on B. Unfortunately in our case the set $\mathbb{N}$ is not an open subset of $\mathbb{C}$, and there are an infinite number of holomorphic functions equal to the factorial function, which are not equal on $\mathbb{R}$ or $\mathbb{C}$. Consider for example the family of functions $\Gamma(z)(1+\sin(k\pi z))$, $k \in \mathbb{Z}$. This set of functions are all equal to the gamma function at integers, but not necessarily equal anywhere else. Each function in this family is a different generalization of the factorial function with analytic properties, what makes Γ so special?

A concise history of the history of Γ is given in [1], here we move forward and discuss various interesting properties of Γ. We have already discussed the recursive property of Γ, that is, $\Gamma(z+1) = z\Gamma(z)$. There are two other known properties of Γ of a similar flavour. The reflection formula is $\Gamma(1-z)\Gamma(z) = \dfrac{\pi}{\sin(\pi z)}$, for $z \notin \mathbb{Z}$, and the multiplication theorem is

$$\prod_{i=0}^{m-1} \Gamma\left(z + \frac{k}{m}\right) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-mz}\Gamma(mz).$$

These properties show how Γ is related to itself. It is reasonable to expect that such relations exist given that the factorial function which began the discussion, is a function with a natural recursive definition. We will prove the reflection property of Γ, but we will first need to establish the identity

$$\sin(\pi z) = \pi z \prod_k \left(1 - \frac{z^2}{k^2}\right), \ z \notin \mathbb{Z}. \tag{4}$$

We follow the approach detailed in [3][page 142], giving more details at times.

Proof. We will begin by establishing the equality

$$\pi\cot(\pi z) = \sum_{k=-\infty}^{\infty} \frac{1}{z+k} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2}. \tag{5}$$

In order to do so, we first show that both $\pi\cot(\pi z)$ and $\displaystyle\sum_{k=-\infty}^{\infty}\frac{1}{z+k}$ satisfy the properties

$$F(z+1) = F(z), \ z \notin \mathbb{Z}, \quad \text{or "}F\text{ is periodic"}, \tag{6}$$

$$F(z) = \frac{1}{z} + F_0(z), \ \text{where } F_0 \text{ is some function, analytic around 0}, \tag{7}$$

$$F \text{ has only simple poles at the integers.} \tag{8}$$

Since $\pi\cot(\pi z) = \pi\dfrac{\cos(\pi z)}{\sin(\pi z)}$, we have (6) because both $\sin(\pi z)$ and $\cos(\pi z)$ are periodic. Since $\cos(\pi z)$ is entire, and $\sin(\pi z)$ entire with zeroes of multiplicity 1 integers and no other zeroes, we have (7) and (8).

For the sum $\displaystyle\sum_{k=-\infty}^{\infty}\frac{1}{z+k}$, notice property (6) is just a shift of the terms, let $\ell = k+1$, then

$$\sum_{k=-\infty}^{\infty} \frac{1}{(z+1)+k} = \sum_{\ell=-\infty}^{\infty}\frac{1}{z+\ell}.$$

The precise argument for (6) is given in the reference. Looking at the sum $\displaystyle\sum_{k=-\infty}^{\infty}\frac{1}{z+k}$, we see the only poles are the simple poles at integer points, so we have property (8). Moreover, since $\displaystyle\sum_{k=1}^{\infty}\frac{2z}{z^2-k^2}$ is analytic in a region near 0, the sum on the left of (5) shows that property (7) is satisfied.

Let

$$\Delta(z) = \pi\cot(\pi z) - \sum_{k=-\infty}^{\infty} \frac{1}{z+k}. \tag{9}$$

Then it is clear that Δ is periodic by property (6) of each term. By property (7) we can see that the singularity at 0 of Δ is removable. Since Δ is periodic, for any bounded neighbourhood S around 0, the neighbourhood $S+1 = \{s+1 \mid s \in S\}$ is a bounded neighbourhood around $s+1$, similarly for $S-1$. Property (8) together with Riemann's theorem on removable singularities implies that all the singularities are removable. Thus, Δ is an entire function on $\mathbb{C}$.

Next we will show that Δ is bounded in order to apply Louville's theorem. Since Δ is periodic, it is sufficient to show that $\Delta(z)$ is bounded on $|\Re(z)| \leq \frac{1}{2}$. Since Δ is entire, and since the subset of points in $\mathbb{C}$ with $|\Re(z)| \leq \frac{1}{2}$ and $\Im(z) \leq 1$ form a closed and bounded subset of $\mathbb{C}$, we have that Δ is bounded on this set. We will now consider those points z with $|\Re(z)| \leq \frac{1}{2}$ and $\Im(z) > 1$, the argument for $\Im(z) < 1$ is similar and follows from the fact that Δ is an odd function.

Let $z = x + yi$ where $|x| \leq \frac{1}{2}$ and $y > 1$. We have

$$\begin{aligned}
\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} &= i\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \\
&= i\frac{e^{i\pi x}e^{-\pi y} + e^{-i\pi x}e^{\pi y}}{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}} \\
&= i\frac{e^{i\pi x}e^{\pi y}e^{-2\pi y} + e^{i\pi x}e^{\pi y}e^{-2i\pi x}}{e^{i\pi x}e^{\pi y}e^{-2\pi y} - e^{i\pi x}e^{\pi y}e^{-2i\pi x}} \\
&= i\frac{e^{-2\pi y} + e^{-2i\pi x}}{e^{-2\pi y} - e^{-2i\pi x}}.
\end{aligned}$$

Using the fact that $e^{-2\pi y}$ is a monotonically decreasing function bounded by 0 and $\frac{1}{e^{2\pi}}$ for $y > 1$, we may take absolute value on each side to obtain

$$\begin{aligned}
|\cot(\pi z)| &= \frac{|e^{-2\pi y} + e^{-2i\pi x}|}{|e^{-2\pi y} - e^{-2i\pi x}|} \\
&\leq \frac{2}{|e^{-2\pi y} - e^{-2i\pi x}|} \\
&\leq \frac{2}{\min\{|e^{-2\pi y} - e^{-2i\pi x}| \ : \ |x| \leq \frac{1}{2}, y > 1\}}.
\end{aligned}$$

Since $0 < e^{-2\pi y} \leq \dfrac{1}{e^{2\pi}}$ lies on the real line and $|e^{-2i\pi x}| = 1$, we can see geometrically

$$\min\{|e^{-2\pi y} - e^{-2i\pi x}|\} \geq \left|\frac{1}{e^{2\pi}} - 1\right|.$$

Thus $|\cot(\pi z)| \leq \dfrac{2}{1 - e^{-2\pi}}$ for $\Im(z) > 1$. On the other hand, the sum

$$\begin{aligned}
\left|\frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}\right| &\leq \left|\frac{1}{x+yi}\right| + \left|\sum_{k=1}^{\infty} \frac{2(x+yi)}{(x+yi)^2 - k^2}\right| \\
&\leq 1 + \sum_{k=1}^{\infty} \frac{|2x + 2yi|}{|x^2 - y^2 + 2ixy - k^2|} \\
&\leq 1 + 2\sum_{k=1}^{\infty} \frac{y}{|y^2 + k^2 - x^2 - 2ixy|}.
\end{aligned}$$

Since $|x^2 - y^2 + 2ixy - k^2| = \sqrt{(y^2 + k^2 - x^2)^2 + 4x^2y^2} \geq y^2 + k^2$, and since $\dfrac{y}{y^2 + t^2}$ is a monotonically decreasing function of t for positive real inputs, we have

$$\sum_{k=1}^{\infty} \frac{y}{|y^2 + k^2 - x^2 - 2ixy|} \leq \sum_{k=1}^{\infty} \frac{y}{y^2 + k^2} \leq \int_0^{\infty} \frac{y\,dt}{y^2 + t^2} = \left[\tan^{-1}\frac{t}{y}\right]_0^{\infty} \leq \frac{\pi}{2}.$$

Therefore by periodicity of Δ for $\Im(z) > 1$ we have

$$|\Delta(z)| \leq |\pi\cot(\pi z)| + \left|\frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}\right| \leq \frac{2\pi}{1 - e^{-2\pi}} + 1 + \pi. \tag{10}$$

Thus Δ is an entire function bounded on all of $\mathbb{C}$. By Liouville's theorem Δ is constant, however since Δ is odd, we must have $\Delta = 0$ and thus (5) is proved.

Let $P(z) = z\prod_k \left(1 - \dfrac{z^2}{k^2}\right)$, and note that

$$\frac{P'}{P}(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}.$$

Also, let $G(z) = \dfrac{\sin(\pi z)}{\pi}$, then $\dfrac{G'}{G} = \pi \cot(\pi z)$. We have by (5) the identity $\frac{P'}{P} = \frac{G'}{G}$, from here we can differentiate

$$\begin{aligned}\left(\frac{P}{G}\right)' &= \frac{PG' - G'P}{G^2} \\ &= \frac{P'G}{G^2} - \frac{G'P}{G^2} \\ &= \frac{P'}{G} - \frac{P'}{G} \\ &= 0.\end{aligned}$$

Thus $P = cG$, we divide may divide both sides of this equality by z and evaluate the limit $z \to 0$, we have

$$\lim_{z\to 0} \frac{\sin(\pi z)}{\pi z} = c \lim_{z\to 0} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right), \tag{11}$$

and thus $c = 1$. Therefore (4) holds. □

We are now prepared to prove the reflection formula,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{12}$$

Proof. We will use the product representation (1) of Γ to prove the formula. We have

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= -z\Gamma(z)\Gamma(-z) \\ &= \frac{z}{z^2}\left(\prod_{k=1}^{\infty} \frac{(1+\frac{1}{k})^z}{1+\frac{z}{k}}\right)\left(\prod_{k=1}^{\infty} \frac{(1+\frac{1}{k})^{-z}}{1+\frac{-z}{k}}\right) \\ &= \frac{1}{z}\left(\prod_{k=1}^{\infty} \frac{(1+\frac{1}{k})^z}{1+\frac{z}{k}} \frac{(1+\frac{1}{k})^{-z}}{1+\frac{-z}{k}}\right) \\ &= \frac{1}{z}\left(\prod_{k=1}^{\infty} \frac{1}{1+\frac{z^2}{k^2}}\right),\end{aligned}$$

here we can apply (4) to obtain (12) for $z \notin \mathbb{Z}$, as desired. □

The reflection formula is a useful property of Γ, for example, we can use it to calculate $\Gamma(\frac{1}{2})$. We have

$$\Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(\frac{1}{2}\Big) = \frac{\pi}{\sin\frac{\pi}{2}} = \pi,$$

since we know Γ is positive for positive real inputs, we have the amazing fact $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. We can also use the reflection formula to evaluate Γ at negative numbers for which formula (2) is not useful.

It has been common to study the function $\frac{1}{\Gamma}$, otherwise known as the reciprocal gamma function, in place of Γ. Interestingly, the reciprocal gamma function is an entire function with only simple zeroes at the non positive integers. It has its own unique characteristics, and in some respects it is a simpler function than Γ. Here we are particularly interested deriving the formula

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}, \tag{13}$$

due to Karl Weierstrass. The number γ is known as the Euler-Mascheroni constant and is defined to be the difference between $\displaystyle\sum_{k\le x} \frac{1}{n}$ and $\displaystyle\int_1^x \frac{dt}{t}$ as $x \to \infty$. Note that each limit when taken individually does not exist, however we will prove that

$$\gamma = \lim_{x\to\infty} \left(\sum_{k\le x} \frac{1}{k} - \int_1^x \frac{dt}{t}\right) \tag{14}$$

exists.

Proof. We begin by showing that $f(x) = \displaystyle\sum_{k\le x} \frac{1}{k} - \int_1^x \frac{dt}{t}$ is bounded below by 0 for $x \in \mathbb{R}^+$. Since $\frac{1}{t}$ is monotonically decreasing on $\mathbb{R}^+$ we have for $P = \{1, 2, \ldots, [x], x\}$ the inequality

$$\int_1^x \frac{dt}{t} \le U(P, f) = \sum_{k\le x} \frac{1}{k}.$$

when $x > [x]$. It is easy to see a similar inequality holds for $x = [x]$, Thus $f \ge 0$ on $\mathbb{R}^+$. Finally, let $\delta \in [0, 1)$, since $\lim_{n\to\infty} \log(1 + \frac{1}{n}) = 0$ we have for any $\epsilon > 0$ the inequality

$$\left|\frac{1}{n} - \log n - \left(\frac{1}{n} - \log(n+\delta)\right)\right| \le \log\left(1 + \frac{1}{n}\right) < \epsilon \text{ for } n > m_1(\epsilon). \tag{15}$$

Let $\gamma_n = f(n)$, we will show

$$\lim_{n\to\infty} \gamma_n = \lim_{x\to\infty} f(x),$$

but first we must know that $\gamma = \lim_{n\to\infty} \gamma_n$ exists. Since f is bounded below, is is clear that γ_n is also bounded below. Moreover, γ_n is a decreasing sequence since

$$\gamma_{n+1} - \gamma_n = \log \frac{n+1}{n} - \frac{1}{n+1}$$

and $\frac{d}{dx}\Big(\log \frac{x+1}{x} - \frac{1}{x+1}\Big) = \frac{-1}{x(x+1)^2}$ is negative on $\mathbb{R}^+$. Thus the limit exists and for any $\epsilon > 0$, we have the inequality

$$|\gamma_n - \gamma| < \epsilon \text{ when } n > m_2(\epsilon). \tag{16}$$

Let $\epsilon > 0$ and let $m = 1 + \max\{m_1(\epsilon/2), m_2(\epsilon/2)\}$, so that $x > m \Rightarrow [x] > \max\{m_1(\epsilon/2), m_2(\epsilon/2)\}$. By the inequalities (16) and (15) we have for $x > m$ the inequality

$$\begin{aligned}
|f(x) - \gamma| &= |f([x] + \{x\}) - \gamma| \\
&= |f([x] + \{x\}) - f([x]) + f([x]) - \gamma| \\
&\leq |f([x] + \{x\}) - f([x])| + |f([x]) - \gamma| \\
&= \Big|\frac{1}{[x]} - \log[x] - \Big(\frac{1}{[x]} - \log([x] + \{x\})\Big)\Big| + |\gamma_{[x]} - \gamma| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

Thus we have established (14). □

The constant γ is a deeply studied constant in mathematics, it occurs in many equations and formulas throughout number theory and analysis. Despite this, there are still many open problems surrounding the constant γ, for example, it is still unknown whether or not γ is rational, despite overwhelming computational evidence that it is not. The connection to Γ by formula (13) is just one of many fascinating appearances. We will now derive formula (13) for positive reals by linking the right hand side to the product representation (1).

Proof. We denote the n'th *harmonic number* by $H_n = \sum_{k\leq n} \frac{1}{k}$ and the function e^x by $\exp(x)$. Here we will also assume that

$$\lim_{n\to\infty} \prod_{k=1}^{n} \Big(1 + \frac{z}{k}\Big) e^{-z/k}$$

exists for positive real numbers with out proof, so that (13) can be written as

$$\begin{aligned}
ze^{\gamma z} \prod_{k=1}^{\infty} \Big(1 + \frac{z}{k}\Big) e^{-z/k} &= z \exp\Big(z \lim_{n\to\infty} H_n - \log n\Big) \lim_{n\to\infty} \prod_{k=1}^{n} \exp(-z/k)\Big(1 + \frac{z}{k}\Big) \\
&= z \lim_{n\to\infty} \exp(z(H_n - \log n)) \prod_{k=1}^{n} \exp(-z/k)\Big(1 + \frac{z}{k}\Big).
\end{aligned} \tag{17}$$

By applying exponential rules, we have

$$\prod_{k=1}^{n} \exp(-z/k) = \exp\Big(-z \sum_{k\leq n} \frac{1}{k}\Big) = \exp(-zH_n).$$

Thus (17) simplifies to

$$\begin{aligned}
z \lim_{n\to\infty} \exp(z(H_n - \log n)) \prod_{k=1}^{n} \exp(-z/k)\Big(1 - \frac{z}{k}\Big) &= z \lim_{n\to\infty} \exp(z(H_n - \log n)) \exp(-zH_n) \prod_{k=1}^{n} \Big(1 + \frac{z}{k}\Big) \\
&= z \lim_{n\to\infty} \exp(-z \log n) \prod_{k=1}^{n} \Big(1 + \frac{z}{k}\Big) \\
&= z \lim_{n\to\infty} n^{-z} \prod_{k=1}^{n} \Big(1 + \frac{z}{k}\Big).
\end{aligned}$$

We can write $n = \prod_{k=1}^{n-1} \frac{k+1}{k} = \prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)$ and substitute to obtain,

$$\begin{aligned} z \lim_{n\to\infty} n^{-z} \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) &= z \lim_{n\to\infty} \left(\prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)\right)^{-z} \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) \\ &= z \lim_{n\to\infty} \prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)^{-z} \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) \\ &= z \prod_{k=1}^{\infty} \frac{1 + \frac{z}{k}}{\left(1 + \frac{1}{k}\right)^{z}} \\ &= \frac{1}{\Gamma(z)}. \end{aligned}$$

□

A more careful and rigorous treatment can show that (13) holds for all $z \in \mathbb{C}$, and complete proof dependant on the Hadamard factorization theorem is given in [3][page 167]. Here we wish to move on to the final section of our discussion.

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n \leq s} \frac{1}{n^s} \text{ for } \Re(s) > 1.$$

It is common to use the letter s as the variable when discussing the ζ function as a function of a complex variable. Today, the function ζ is a famous function in mathematics largely due to a long standing unsolved problem known as the *Riemann Hypothesis.* Aside from this however, the function zeta has its own rich history and deep theory. An interesting equation to note is the functional equation connecting Γ and ζ discovered by Riemann in the 19th century,

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s). \tag{18}$$

The functions Γ and ζ are two of the most significant known non elementary functions in mathematics, as a result equations such as (18) are heavily studied. Here we will prove the identity

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty t^{s/2-1} \frac{\vartheta(t) - 1}{2} dt \tag{19}$$

involving Γ, ζ and $\vartheta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}$.

Proof. By the substitution $u = \frac{t}{n^2\pi}$, we have the identity

$$\int_0^\infty t^{s/2-1} e^{-n^2\pi t} dt = \frac{1}{\pi^{s/2} n^s} \int_0^\infty u^{s/2-1} e^{-u} du = \frac{\Gamma(s/2)}{\pi^{s/2} n^s}. \tag{20}$$

The function ϑ is connected to the integrand on the left hand side of (20) by

$$\frac{\vartheta(x) - 1}{2} = \sum_{n=1}^{\infty} e^{-n^2\pi x}.$$

Let $\psi = \frac{\vartheta - 1}{2}$, we will need to show that the sequence of functions

$$\psi_k(x) = \sum_{n=1}^{k} e^{-n^2\pi x}$$

is uniformly convergent on $[a, \infty)$, for $a \in \mathbb{R}^+$ in order to continue. Let $\epsilon > 0$, $a \in \mathbb{R}^+$. We have

$$\begin{aligned} |\psi(x) - \psi_k(x)| &= \left| \sum_{n=1}^{\infty} e^{-n^2\pi x} - \sum_{n=1}^{k} e^{-n^2\pi x} \right| \\ &= \sum_{n=k+1}^{\infty} e^{-n^2\pi x} \\ &\leq \sum_{n=k+1}^{\infty} e^{-n\pi x} \\ &\leq \sum_{n=k+1}^{\infty} e^{-n\pi a} \end{aligned}$$

Note that $\sum_{n=k+1}^{\infty} e^{-n^2\pi a}$ is the tail of a geometric series with $r = \frac{1}{e^{\pi a}}$, therefore we have

$$\begin{aligned}\sum_{n=k+1}^{\infty} e^{-n\pi a} &= \frac{1}{1-e^{-\pi a}} - \sum_{n=0}^{k}\left(\frac{1}{e^{\pi a}}\right)^n \\ &= \frac{1}{1-e^{-\pi a}} - \frac{1-e^{-(k+1)\pi a}}{1-e^{-\pi a}} \\ &= e^{-k\pi a}\frac{e^{-\pi a}}{1-e^{-\pi a}}.\end{aligned}$$

Since $\lim_{k\to\infty} e^{-k\pi a}\dfrac{e^{-\pi a}}{1-e^{-\pi a}} = 0$ when $a \in \mathbb{R}^+$ is fixed, there exists $m(\epsilon, a)$ such that $k > m(\epsilon, a)$ implies

$$e^{-k\pi a}\frac{e^{-\pi a}}{1-e^{-\pi a}} < \epsilon.$$

Thus, for $x \in [a, \infty)$ and $k > m(\epsilon, a)$ we have

$$|\psi(x) - \psi_k(x)| < \epsilon.$$

Therefore the functions ψ_k are uniformly convergent to ψ, and we may interchange summation and integration to obtain

$$\begin{aligned}\int_0^{\infty} t^{s/2-1}\sum_{n=1}^{\infty} e^{-n^2\pi t}dt &= \sum_{n=1}^{\infty}\int_0^{\infty} t^{s/2-1}e^{-n^2\pi t}dt \\ &= \sum_{n=1}^{\infty}\frac{\Gamma(s/2)}{\pi^{s/2}n^s} \\ &= \frac{\Gamma(s/2)\zeta(s)}{\pi^{s/2}}\end{aligned}$$

as desired. □

The function ϑ is a belongs to a family of functions known as Jacobi theta functions. The function ϑ satisfies the functional equation $\vartheta(x) = x^{-1/2}\vartheta(1/x)$, from which we may derive another functional equation involving Γ and ζ as is done in chapter 6 of [3]. Beyond what we have discussed in this report, there are still many more interesting things about Γ left to study that we have yet to mention, logarithmic convexity and the Beta function to name a few which are concisely covered in [2]. However, these topics will have to wait for the next report on Euler's Gamma Function.

References

[1] Philip J. Davis, *Leonhard Euler's Integral: An Historical Profile of the Gamma Function*, The American Mathematical Monthly, 1959, pp. 849-869.

[2] Emil Artin, *The Gamma Function*, translated by Michael Butler, 2015.

[3] Elias M. Stein and Rami Shakarchi, *Complex Analysis*, 2007.